The Nash equilibrium in rock paper scissors

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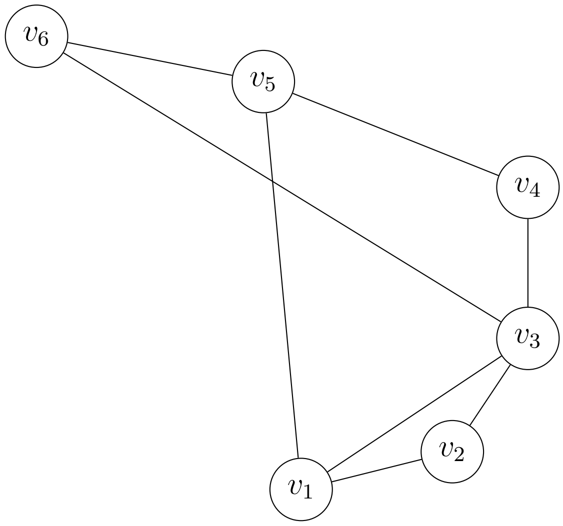
1 Problem Description

Imagine playing multiplayer rock paper scissors (RPS) on a graph, where each vertex is a player, and they are playing against all of their neighbors. Each vertex is only allowed to play one strategy, and each edge represents a game of RPS giving payoffs to players on each end. The utility of a vertex is the sum of the payoffs of all the games that they play with their neighbors. For example, the following graph:



Formally, let $G = (V, E)$ be a graph with at least one edge. The vertices $V = \{1, \cdots, n\}$ are the players. Each player has the same pure strategy set $T = \{R, P, S\}$. For each edge vw, we define the utility of v and w when playing pure strategies s_v and s_w by $u_v^w(s_v, s_w)$ and $u_w^v(s_w, s_v)$, and this utility is summarized in the following table:

P_1 \ P_2	R	P	S
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

Given a pure strategy profile s, the utility of a player v is then

$$u_v(s) = \sum_{vw \in E} u_v^w(s_v, s_w)$$

Let x be a mixed strategy profile, where x^v denotes the mixed strategy (x_R^v, x_P^v, x_S^v) played by vertex v. Then the expected utility of player v is

$$u_v(x) = \sum_{vw \in E} \sum_{s_v, s_w \in T} u_v^w(s_v, s_w) x_{s_v}^v x_{s_w}^w$$

We let Δ be the set of all mixed strategy profiles. We aim to find the Nash equilibrium in general graphs.

2 Zero-sum property

- **Break up**: The entire game can be viewed as a collection of sub-games, where each sub-game is a duel between two neighboring players. The outcome of each sub-game only affects the payoffs of the two players involved, and each player's total utility is the sum of the payoffs from all games they play with their neighbors.

- **Claim**: We claim that every sub-game is a zero-sum game. That is, a rock-paper-scissors game between two players is a zero-sum game.

- **Proof of Claim**:

  Proof. Recall that a two-player strategic game is zero-sum if

  $$\forall x \in \Delta, u_1(x) + u_2(x) = 0$$

  where Δ is the set of all possible strategy profile.

  - We first consider the pure strategy profile $x \in \Delta$:
    * If $u_1(x) = 1$, this means player 1 won, player 2 lost, so $u_2(x) = -1$.
    * If $u_2(x) = 0$, this means tie, so $u_2(x) = 0$.
    * If $u_1(x) = -1$, this means player 1 lost, player 2 won, so $u_2(x) = 1$.

    All cases satisfy $u_1(x) + u_2(x) = 0$, so this is a zero-sum game under pure strategy.
  - Then consider the mixed strategy profile $x = ((x_R^1, x_P^1, x_S^1), (x_R^2, x_P^2, x_S^2)) \in \Delta$

    $$\begin{aligned} u_1(x) &= x_R^1 \cdot (0 \cdot x_R^2 + (-1) \cdot x_P^2 + 1 \cdot x_S^2) \\ &+ x_P^1 \cdot (1 \cdot x_R^2 + 0 \cdot x_P^2 + (-1) \cdot x_S^2) \\ &+ x_S^1 \cdot ((-1) \cdot x_R^2 + 1 \cdot x_P^2 + 0 \cdot x_S^2) \\ &= x_R^1 x_S^2 + x_P^1 x_R^2 + x_S^1 x_P^2 - (x_P^1 x_S^2 + x_S^1 x_R^2 + x_R^1 x_P^2) \end{aligned}$$

    $$\begin{aligned} u_2(x) &= x_R^2 \cdot (0 \cdot x_R^1 + (-1) \cdot x_P^1 + 1 \cdot x_S^1) \\ &+ x_P^2 \cdot (1 \cdot x_R^1 + 0 \cdot x_P^1 + (-1) \cdot x_S^1) \\ &+ x_S^2 \cdot ((-1) \cdot x_R^1 + 1 \cdot x_P^1 + 0 \cdot x_S^1) \\ &= -(x_R^1 x_S^2 + x_P^1 x_R^2 + x_S^1 x_P^2) + (x_P^1 x_S^2 + x_S^1 x_R^2 + x_R^1 x_P^2) \\ &= -u_1(x) \end{aligned}$$

    $u_1(x) + u_2(x) = 0$ holds, this is a zero-sum game under mixed strategy.

  □

- **Conclusion**: We've proved each sub-game between two neighboring players is a zero-sum game, and since the entire game consists of multiple such zero-sum sub-games, the gains and losses across all players cancel each other out, resulting in a total utility sum of zero. Therefore, this is a zero-sum game.

- **Another approach**: We can also compute the result directly.

$$\begin{aligned}
\sum_{v\in V} u_v(x) &= \sum_{v\in V}\sum_{vw\in E}\sum_{s_v,s_w\in T} u_v^w(s_v,s_w)x_{s_v}^v x_{s_w}^w \\
&= \sum_{vw\in E}\sum_{s_v,s_w\in T} (u_v^w(s_v,s_w)+u_w^v(s_v,s_w))x_{s_v}^v x_{s_w}^w \\
&= \sum_{vw\in E}\sum_{s_v,s_w\in T} (u_v^w(s_v,s_w)+(-u_v^w(s_v,s_w)))x_{s_v}^v x_{s_w}^w \\
&= \sum_{vw\in E}\sum_{s_v,s_w\in T} (0)x_{s_v}^v x_{s_w}^w \\
&= \sum_{vw\in E}\sum_{s_v,s_w\in T} (0) \\
&= 0
\end{aligned}$$

  Note that we used the claim that a rock-paper-scissors game between two players is a zero-sum game, which is proved on last page. Now we can conclude that this game is a zero-sum game.

3 A Linear Program

Consider the following Linear Program, we will use this to find the Nash equilibrium.

$$\begin{aligned}
\min\ & \sum_{v\in V} z_v \\
s.t.\ & z_v \geq u_v(s, y_{-v}) \quad \forall v\in V, s\in T \\
& y_R^v + y_P^v + y_S^v = 1 \quad \forall v \in V \\
& y \geq \mathbb{0}
\end{aligned}$$

We claim that the optimal value of (P) must be non-negative.

Proof. We have

$$\begin{aligned}
\sum_{v\in V} z_v &= \sum_{v\in V}\max_{s\in T} u_v(s, y_{-v}) \\
&\geq \sum_{v\in V} u_v(y^v, y_{-v}) \\
&= \sum_{v\in V} u_v(y) = 0
\end{aligned}$$

as it is a zero-sum game we've proved. Therefore, the optimal value of (P) must ≥ 0. □

By Nash's theorem, there exists a Nash equilibrium y^*, such that no player could improve their utility by changing their strategies. That is,

$$\forall v \in V, z_v = \max_{s\in T} u_v(s, y_{-v}^*) = u_v(y_v^*, y_{-v}^*) = u_v(y^*)$$

Therefore, as it is a zero-sum game, the object value is:

$$\sum_{v\in V} z_v = \sum_{v\in V} u_v(y^*) = 0$$

We've proved the optimal value of (P) must ≥ 0, and this is a minimizing program. Therefore, we can conclude that pair (y^*, z) is also the optimal solution.

We now claim that if (y, z) is an optimal solution to (P), then y is a Nash equilibrium.

Proof. Assume the hypothesis, (y, z) is an optimal solution to (P). As we've proved, the optimal value of (P) is 0, therefore,

$$\sum_{v \in V} z_v = 0$$

Assume by contradiction, y is not a Nash equilibrium. Therefore, there exists a player (say m) who can improve the utility by changing the strategy. Therefore,

$$z_m \geq \max_{s \in T} u_m(s, y_{-m}) > u_m(y)$$

So by changing player m's strategy, the object value could be

$$\sum_{v \in V} z_v = \underbrace{0}_{\text{previous object value}} - \max_{s \in T} u_m(s, y_{-m}) + u_m(y) < 0$$

This leads to a contradiction since we've proved that the optimal value is 0, and this is a minimizing problem, it is impossible to have an objective value less than 0 while the optimal value is 0. Therefore, we can conclude that if (y, z) is an optimal solution to (P), then y is a Nash equilibrium. □

Now we can finally back to topic: developing a method to find the Nash equilibrium. From the previous steps we found that in the optimal solution (y, z), y is a Nash equilibrium. Therefore, we can find the Nash equilibrium by finding the optimal solutions to a linear program. Here's the general form of our program that need to be solved:

$$\begin{aligned} \min \quad & \sum_{v \in V} z_v \\ s.t. \quad & z_v \geq u_v(s, y_{-v}) \quad \forall v \in V, s \in T \\ & y_R^v + y_P^v + y_S^v = 1 \;\; \forall v \in V \\ & y \geq \mathbb{0} \end{aligned}$$

By some transformations, we can turn it into the following linear program:

$$\begin{aligned} \min \quad & \sum_{v \in V} z_v \\ s.t. \quad & \left(z_v \geq \begin{cases} y_S^u - y_P^u, \\ y_R^u - y_S^u, \\ y_P^u - y_R^u \end{cases} \right) \quad \forall v \in V, vu \in E \\ & y_R^v + y_P^v + y_S^v = 1 \;\; \forall v \in V \\ & y \geq \mathbb{0} \end{aligned}$$

By plugging in the variables, and apply Simplex method, we can find an optimal solution. But we can use another way, by claiming a feasible solution be optimal, and then verify the object value optimal. Now we claim the feasible solution optimal for all graphs:

$$z = \mathbb{0} \text{ and } \forall v \in V, y_R^v = y_P^v = y_S^v = 1/3$$

We first verify if the constraints still hold:

- From $\left(z_v \geq \begin{cases} y_S^u - y_P^u, \\ y_R^u - y_S^u, \\ y_P^u - y_R^u \end{cases} \right) \forall v \in V, vu \in E$, we obtained several $\left(0 \geq \begin{cases} 0, \\ 0, \\ 0 \end{cases} \right)$.
- From $y_R^v + y_P^v + y_S^v = 1 \forall v \in V$, we obtained several $1/3 + 1/3 + 1/3 = 1$.
- From $y \geq \mathbb{0}$, we obtained several $1/3 \geq 0$.

Therefore, all constraints are satisfied, and the object value is equal to 0, which is optimal as we proved. Also, if (y, z) is an optimal solution to (P), then y is a Nash equilibrium. Therefore,

$$\forall v \in V, y_R^v = y_P^v = y_S^v = 1/3$$

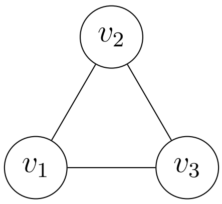
is a Nash equilibrium.

4 Discussion

From our previous steps, we successfully find that

$$\forall v \in V, y_R^v = y_P^v = y_S^v = 1/3$$

is a Nash equilibrium, but the fact is this Nash equilibrium is not unique. Consider:



Note v_1, v_2, v_3 sequentially play Rock, Paper, Scissors is a Nash equilibrium.

5 Conclusion

This study investigated the multiplayer rock-paper-scissors game on graph structures and analyzed its zero-sum properties. By employing linear programming methods, we successfully identified the Nash equilibrium for all players, revealing that each player's strategy converges to a uniform distribution. This result not only offers a new perspective on game theory but also provides a theoretical foundation for practical applications. While we identified one Nash equilibrium, it is important to note that this equilibrium is not unique; future research could further explore other potential equilibrium states and their behaviors in complex networks. Overall, this study contributes valuable insights into the understanding of multiplayer games and lays the groundwork for future research directions.

6 Reference

1. Martin Pei, "Game Theory Assignments 3". [private access]